SOME RECENT PROGRESS IN HIGHER KOSZULITY

YU YE, PU ZHANG

Dedicated to Professor Ke-Qin Feng on the occasion of his 70th birthday

Abstract. In this article, we outline necessary backgrounds, main definitions, basic results, and some recent progress, in the theory of higher Koszul algebras and modules.

Koszuality has been introduced by Priddy [Pr]. It reflects a fundamental feature of graded objects of study. Koszul duality developed by Beilinson, Ginzburg and Soergel [BGS] reveals the relations between a Koszul algebra and its quadratic dual, not only in the level of the algebraic structures, but also in the one of their derived categories.

Ten years ago, higher Koszulity was introduced by Berger [Ber1] in the connected case, and by Green et al. [GMMZ] in the nonconnected case. In the recent development, it turns out to be important, for example in the representation theory of algebras, noncommutative geometry, Artin-Schelter algebras, Calabi-Yau algebras, and Yang-Mills algebras (see e.g. [Ber1], [Ber3], [BD], [BT], [Boc], [CD], [EP], [GMMZ], [LPWZ], [MS1], [V2]).

The aim of this article is to outline basic results and some recent progress, especially some contributions by Chinese mathematicians, in the theory of higher Koszul algebras and modules. For the convenience of the reader, we also include necessary preliminaries and backgrounds, the main definitions, remarks and questions in this direction.

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1. Preliminaries

Throughout Λ is a **standardly graded algebra** over a field k ([GMV]), i.e., $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ is a positively graded k-algebra satisfying

(*i*) $\Lambda_0 = k^r$ for some integer $r \geq 1$;
(*ii*) $\dim_k \Lambda_i < \infty$, $\forall\ i \geq 0$; and

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Supported by the NSFC (10725104 and 10971206).
yeyu@ustc.edu.cn, pzhang@sjtu.edu.cn.

(*iii*) $\Lambda_i \Lambda_j = \Lambda_{i+j}, \ \forall \ i, j \geq 0$.

A **graded left Λ-module** M is a left Λ-module with a decomposition of k-spaces $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that $\Lambda_i M_j \subseteq M_{i+j}, \ \forall \ i, j \in \mathbb{Z}$. For graded Λ-modules M and N, a Λ-morphism $f\colon M \to N$ is a **graded morphism** if $f(M_i) \subseteq N_i, \ \forall \ i \in \mathbb{Z}$. Let $\mathrm{Mod}(\Lambda)$ be the category of left Λ-modules, and $\mathrm{Gr}(\Lambda)$ the category of left graded Λ-modules and graded morphisms. They are both abelian categories. Let $\mathrm{mod}(\Lambda)$ (*resp.* $\mathrm{gr}(\Lambda)$) be the full subcategory of $\mathrm{Mod}(\Lambda)$ (*resp.* $\mathrm{Gr}(\Lambda)$) consisting of finitely generated Λ-modules. They are both abelian categories if Λ is noetherian. Thus, for example, $\mathrm{gr}(\Lambda^e)$ is the category of finitely generated graded Λ-Λ-bimodules, here Λ^{op} is the opposite algebra of Λ, and $\Lambda^e = \Lambda \otimes_k \Lambda^{\mathrm{op}}$ is the enveloping algebra of Λ.

Let S be a subset of $\mathbb{Z}$. A graded Λ-module M is **generated in degrees in** S if $M = \Lambda(\bigoplus_{s \in S} M_s)$; M is **generated in degree** s if $M = \Lambda M_s$; M is **supported above degree** n if $M_j = 0$ for $j < n$; M is **bounded below** if M is supported above degree n for some n; and M is **concentrated in degrees in** S if $M_i = 0$ for $i \notin S$; and finally, M is **locally finite** if $\dim_k(M_i) < \infty$ for all $i \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, the **shift** functor (n) on $\mathrm{Gr}(\Lambda)$ is defined by $M(n)_i = M_{i-n}$ for $M \in \mathrm{Gr}(\Lambda)$.

The regular module ${}_\Lambda\Lambda$ is a projective object in $\mathrm{Gr}(\Lambda)$ and in $\mathrm{gr}(\Lambda)$; and conversely, any projective object in $\mathrm{Gr}(\Lambda)$ and in $\mathrm{gr}(\Lambda)$ is a direct summand of a direct sum $\bigoplus_{s \in S} \Lambda(n_s)$, here S is an index set and $n_s \in \mathbb{Z}$ for each s. The **graded Jacobson radical** $J(\Lambda)$ of Λ is the ideal $\bigoplus_{i \geq 0} \Lambda_i$. The trivial Λ-module $\Lambda_0 = \Lambda / J(\Lambda)$ is a graded semi-simple Λ-module concentrated in degree 0.

Denote by $\mathrm{E}(\Lambda)$ the Ext-algebra $\bigoplus_{i \geqslant 0} \mathrm{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$ which is positively graded, with multiplication given by the Yoneda product. Consider the even Ext-algebra $\mathrm{E}^{\mathrm{ev}}(\Lambda) := \bigoplus_{i \geqslant 0} \mathrm{Ext}^{2i}_\Lambda(\Lambda_0, \Lambda_0)$ which is again positively graded with grading $\mathrm{E}^{\mathrm{ev}}(\Lambda)_i := \mathrm{Ext}^{2i}_\Lambda(\Lambda_0, \Lambda_0)$. For $M \in \mathrm{Gr}(\Lambda)$, let $\mathscr{E}(M)$ be the graded $\mathrm{E}(\Lambda)$-module $\bigoplus_{i \geqslant 0} \mathrm{Ext}^i_\Lambda(M, \Lambda_0)$, with action given by the Yoneda product. Consider also the even Ext-module $\mathscr{E}^{\mathrm{ev}}(M) := \bigoplus_{i \geqslant 0} \mathrm{Ext}^{2i}_\Lambda(M, \Lambda_0)$, and the odd Ext-module $\mathscr{E}^{\mathrm{odd}}(M) := \bigoplus_{i \geqslant 0} \mathrm{Ext}^{2i+1}_\Lambda(M, \Lambda_0)$, which are graded $\mathrm{E}^{\mathrm{ev}}(\Lambda)$-modules with gradings

$$\mathscr{E}^{\mathrm{ev}}(M)_i := \mathrm{Ext}^{2i}_\Lambda(M, \Lambda_0), \quad \text{and} \quad \mathscr{E}^{\mathrm{odd}}(M)_i := \mathrm{Ext}^{2i+1}_\Lambda(M, \Lambda_0), \ \forall \ i \geq 0.$$

Let $M \in \mathrm{Gr}(\Lambda)$ be bounded below. Consider $J(M) = J(\Lambda)M = \Lambda_1 M$, the **radical** of M. Since the identity of Λ is in Λ_0, we have $J(M) \neq M$ for $M \neq 0$. For $i \in \mathbb{Z}$, let $M_i = T_i \oplus \Lambda_1 M_{i-1}$ be a decomposition of k-spaces. Then $\Lambda \otimes_k T_i$ is a projective Λ-module generated in degree 0. Put $P_M = \bigoplus_{i \in \mathbb{Z}} (\Lambda \otimes_k T_i)(i)$. Then the epimorphism $\pi\colon P_M \twoheadrightarrow M$, given by the action of Λ on M, is a **projective cover** of M in $\mathrm{Gr}(\Lambda)$ in the sense that π is a direct summand of any other graded epimorphism from a graded projective Λ-module to M. This shows that if $M \in \mathrm{Gr}(\Lambda)$ is bounded below (in particular, if $M \in \mathrm{gr}(\Lambda)$), then M admits a minimal graded projective resolution

$$Q^\bullet\colon \ \cdots \longrightarrow Q^2 \longrightarrow Q^1 \longrightarrow Q^0 \longrightarrow M, \tag{1}$$

where each Q^i is bounded below. If each Q^i is finitely generated, then we say that M admits **a finitely generated graded projective resolution**.

Let GHom_Λ and GExt^i_Λ denote the morphisms and extensions in $\mathrm{Gr}(\Lambda)$, as opposed to the usual Hom_Λ and Ext^i_Λ in $\mathrm{Mod}(\Lambda)$. For any $M \in \mathrm{gr}(\Lambda)$, then for each $N \in \mathrm{Gr}(\Lambda)$ we have

$$\mathrm{Hom}_\Lambda(M,N) = \bigoplus_{n\in Z} \mathrm{GHom}_\Lambda(M, N(n)),$$

i.e., $\mathrm{Hom}_\Lambda(M,N)$ becomes a $\mathbb{Z}$-graded k-space with shift grading

$$\mathrm{Hom}_\Lambda(M,N)_i = \mathrm{GHom}_\Lambda(M, N(i)), \ \forall\ N \in \mathrm{Gr}(\Lambda).$$

Moreover, if M admits a finitely generated graded projective resolution, then for each $N \in \mathrm{Gr}(\Lambda)$ and each $i \geq 0$ we have

$$\mathrm{Ext}^i_\Lambda(M,N) = \bigoplus_{n\in \mathbb{Z}} \mathrm{GExt}^i_\Lambda(M, N(n)),$$

i.e., $\mathrm{Ext}^i_\Lambda(M,N)$ becomes a $\mathbb{Z}$-graded k-space with $\mathrm{Ext}^i_\Lambda(M,N)_n = \mathrm{GExt}^i_\Lambda(M,N(n))$. In particular, $\mathscr{E}(M)$, $\mathrm{E}^{\mathrm{ev}}(M)$ and $\mathrm{E}^{\mathrm{odd}}(M)$ are all bigraded k-spaces and the Yoneda product is always bigraded.

2. Higher Koszulity

The following is one of the central notion in this article.

Definition 2.1. *Let Λ be a standardly graded algebra, and $d \geq 2$ an integer. A finitely generated graded Λ-module M is a d-**Koszul module** if M admits a graded projective resolution* (1) *such that each Q^i is generated in degree $\delta(i)$, where*

$$\delta(i) := \begin{cases} nd, & \text{if } i = 2n, \\ nd+1, & \text{if } i = 2n+1. \end{cases}$$

*If the trivial Λ-module Λ_0 is a d-Koszul module, then we call Λ a d-**Koszul algebra**.*

If $d = 2$ then the d-Koszulity is exactly the (classical) Koszulity introduced in [Pr]. For the theory of Koszulity we also refer to [BGS] and [GMV]. This notion of d-Koszul module is introduced in [Ber1], where Λ is connected in the sense that $\Lambda_0 = k$; and the same notion in general case, is introduced independently in [GMMZ] (however, it is published later for some reasons).

Remark 2.2. (*i*) *If M is a d-Koszul module, then a graded projective resolution $\mathbf{Q}^\bullet$ of M is unique up to isomorphism (see* [BGS]*, p.476), and each Q^i in* (1) *is finitely generated: this is important for the application of the shift grading on $\mathrm{Ext}^n_\Lambda(M,-)$.*

(*ii*) *If M is a d-Koszul module, then a graded projective resolution $\mathbf{Q}^\bullet$ of M is minimal, and each syzygy $\Omega^i(M)$ is a graded Λ-module finitely generated in degree $\delta(i)$. In particular, M is finitely generated in degree 0.*

(iii) A finitely generated graded Λ-module M is d-Koszul if and only if $\mathscr{E}^{2n}(M)$ is concentrated in degree nd, and $\mathscr{E}^{2n+1}(M)$ is concentrated in degree $nd+1$ for each $n \geq 0$, both with the shift grading.

Let $Q^\bullet$ be a minimal graded projective resolution of $M = \Lambda_0$. A direct calculation shows that $\mathrm{Tor}_i^\Lambda(\Lambda_0, \Lambda_0) \cong Q^i/J(Q^i)$. Thus we have the following equivalent definition for d-Koszul algebras, which is exactly the one used in [Ber1, Definition 2.10].

Proposition 2.3. *A standardly graded algebra Λ is d-Koszul if and only if $\mathrm{Tor}_i^\Lambda(\Lambda_0, \Lambda_0)$ is concentrated in degree i for each $i \geq 0$.*

Since $\mathrm{Tor}_i^\Lambda(\Lambda_0, \Lambda_0)$ is left-right symmetric, we have (see [YZ1]).

Proposition 2.4. *A standardly graded algebra Λ is d-Koszul if and only if so is Λ^{op}.*

3. Higher Koszul complexes

Let $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ be a standardly graded algebra. Then we have the tensor algebra $T_{\Lambda_0}(\Lambda_1) = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_1^{\otimes 2} \oplus \cdots$, and $\Lambda \cong T_{\Lambda_0}(\Lambda_1)/I$ for some homogeneous ideal I, where $\otimes = \otimes_{\Lambda_0}$. Set $I_d = I \cap \Lambda_1^{\otimes d}$. If $I = \langle I_d \rangle$ for some $d \geq 2$, then we call Λ a **d-homogeneous**.

Lemma 3.1. *Λ is d-homogeneous if and only if $\mathrm{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)$ is concentrated in degree d.*

An easy consequence is that a d-Koszul algebra is always d-homogeneous. The converse is not true in general. Let $\Lambda \cong T_{\Lambda_0}(\Lambda_1)/I$ be a d-homogeneous algebra and $R = I_d$. Construct the (left) **d-Koszul complex**

$$K^\bullet: \ \cdots \longrightarrow K^2 \xrightarrow{d^2} K^1 \xrightarrow{d^1} K^0 \xrightarrow{\pi} \Lambda_0 \longrightarrow 0$$

of Λ as follows. Put $K_0^0 = \Lambda_0$, $K_1^1 = \Lambda_1$, and for $i \geq 2$,

$$K^i_{\delta(i)} = \bigcap_{0 \leq s \leq \delta(i)-d} \Lambda_1^{\otimes s} \bigotimes R \bigotimes \Lambda_1^{\otimes \delta(i)-d-s}.$$

Then $K^i = \Lambda \otimes K^i_{\delta(i)}$ is a projective module which is generated in degree $\delta(i)$, where $\delta(i)$ is as in Definition 2.1. The differential map $d^i\colon K^i \to K^{i-1}$ is given by

$$d^{2n}(a \otimes (a_1 \otimes \cdots \otimes a_{nd})) = aa_1 \cdots a_{d-1} \otimes (a_d \otimes \cdots \otimes a_{nd}), \ n \geq 1$$

and

$$d^{2n+1}(a \otimes (a_1 \otimes \cdots \otimes a_{nd+1})) = aa_1 \otimes (a_2 \otimes \cdots \otimes a_{nd+1}), \ n \geq 0.$$

The following fundamental result of d-Koszul algebras is proved in [Ber1] for the connected case, and in [GMMZ] for the nonconnected case.

Theorem 3.2. *Let $d \geq 2$ be an integer and Λ a d-homogeneous algebra. Then Λ is d-Koszul if and only if the d-Koszul complex of Λ is a minimal graded projective resolution of Λ_0.*

The "if" part is obvious and for the "only if" part, one uses the following lemma ([GMMZ, Lemma 8.1]).

Lemma 3.3. *Keeping the above notation. If Λ is d-Koszul, then for any $1 \leq n \leq d-1$,*

$$(\Lambda_1^{\otimes n} \otimes R) \bigcap (R \otimes \Lambda_1^{\otimes n}) = \bigcap_{0 \leq s \leq n} \Lambda_1^{\otimes s} \otimes R \otimes \Lambda_1^{\otimes n-s}.$$

We can also construct a d-Koszul complex $B^\bullet = (B^i, b^i)$ of bimodules for a d-homogeneous algebra Λ as follows. Set $B^i = \Lambda \otimes K^i_{\delta(i)} \otimes \Lambda$. Thus $B^0 = \Lambda \otimes \Lambda$ and $B^1 = \Lambda \otimes \Lambda_1 \otimes \Lambda$. The differential $b^i \colon B^i \to B^{i-1}$ is given by

$$\begin{aligned} & b^{2n}(a \otimes (a_1 \otimes \cdots \otimes a_{nd}) \otimes a') \\ = & \sum_{1 \leq s \leq d} a a_1 \cdots a_{s-1} \otimes (a_s \otimes \cdots \otimes a_{s+(n-1)d}) \otimes a_{s+(n-1)d+1} \cdots a_{nd} a', \end{aligned}$$

and

$$\begin{aligned} & b^{2n+1}(a \otimes (a_1 \otimes \cdots \otimes a_{nd+1}) \otimes a') \\ = & a a_1 \otimes (a_2 \otimes \cdots \otimes a_{nd+1}) \otimes a' - a \otimes (a_1 \otimes a_2 \otimes \cdots \otimes a_{nd}) \otimes a_{nd+1} a'. \end{aligned}$$

This construction appear first in [Ber1, Section 5], and in [Y, Section 2.4] for the nonconnected case. The original form given in [Ber1] is not correct, and an erratum is made later by Berger himself.

Now we have another characterization for d-Koszulity by using d-Koszul bimodule complex. Compare to Theorem 2.3 in [XX], where a different argument by using homotopy of complexes is given.

Theorem 3.4. ([Ber1, Theorem 5.6], [Y, Theorem 2.4.2]) *A d-homogeneous algebra Λ is d-Koszul if and only if the d-Koszul bimodule complex $B^\bullet$ combined with the multiplication map $\Lambda \otimes \Lambda \twoheadrightarrow \Lambda$ is a minimal projective resolution of Λ in the category* $\mathrm{gr}(\Lambda^e)$.

An immediate consequence is that for a d-Koszul algebra, the Hochschild homology dimension coincides with the global dimension, see for example [Ber2] and [BM, Theorem 4.5]. One may also use the Koszul complex of bimodules to compute the Hochschild homology and cohomology. In [Ber1, Section 1], the Hochschild homology of a connected d-Koszul algebra whose relations are given by the anti-symmetrizers of degree d is computed. Marconnet [M1, M2] computed the Hochschild homology of cubic AS-regular algebras. Another result worthy mentioning is that the multiplication of Hochschild cohomology rings of a d-Koszul algebra is given explicitly by Xu and Xiang [XX, Theorem 3.2].

4. Hilbert and Poincaré series

Let Λ be a standardly graded algebra with $\Lambda_0 = k^r$, $1 = e_1 + \cdots + e_r$ the identity, and S_i the graded simple module corresponding to e_i. For $l \geq 0$, let H^l be the $r \times r$ matrix with entries $H^l_{ij} = \dim_k(e_i \Lambda_l e_j)$. The **Hilbert series** of Λ is the $r \times r$ matrix

$H(\Lambda, x) = H^0 + H^1 x + H^2 x^2 + \cdots$, with entries in $\mathbb{Z}[[x]]$. Then $H(\Lambda, x)$ is invertible, and the inverse is the **Poincaré series** of Λ, which is denoted by $P(\Lambda, x)$.

Theorem 4.1. ([W, Theorem 1.5]) *One has*

$$P(\Lambda, x)_{ij} = \sum_{u\geq 0}\sum_{v\geq 0}(-1)^v \dim_k(\mathrm{GExt}^v_\Lambda(S_i, S_j(u)))x^u.$$

In [W] Λ is required to be either finite-dimensional, or of finite global dimension. We point out that the argument there works also for a standardly graded algebra. One way to see this is to regard the Grothendieck group of $\mathrm{gr}(\Lambda)$ as a $\mathbb{Z}[[x]]$-module. It has two natural bases: one consists of the isoclasses of simple modules, and another consists of the isoclasses of indecomposable projective modules. Roughly speaking, the Hilbert series and the Poincaré series are nothing but the transformation matrices between these two bases.

One can define the Hilbert series of a graded module in the similar way. Let $M \in \mathrm{Gr}(\Lambda)$ be supported above degree 0 and locally finite. The Hilbert series $H(\Lambda, M, x)$ of M is by definition the column vector with r-entries in $\mathbb{Z}[[x]]$ given by

$$H(\Lambda, M, x)_i = \sum_{u\geq 0} \dim_k(e_i M)x^u.$$

The Poincaré series $P(\Lambda, M, x)$ of M is the row vector with r-entries in $\mathbb{Z}[[x]]$ given by

$$P(\Lambda, M, x)_j = \sum_{u\geq 0}\sum_{v\geq 0}(-1)^v \dim_k(\mathrm{GExt}^v_\Lambda(M, S_j(u)))x^u.$$

We have the following connection between the Poincaré series and the Hilbert series.

Proposition 4.2. *Let Λ be a standardly graded algebra and $M \in \mathrm{Gr}(\Lambda)$ be supported above degree 0 and locally finite. Then $H(\Lambda, M, x) = H(\Lambda, x)P(\Lambda, M, x)$.*

Note that $H(\Lambda, S_i, x)$ is the i-th row of the identity matrix. It follows that the proposition above generalizes Theorem 3.1. As an application, we get the following numerical characterization for d-Koszulity.

Theorem 4.3. *Let Λ be a d-homogeneous algebra and $M \in \mathrm{gr}_0(\Lambda)$. Then M is d-Koszul if and only if*

$$P(\Lambda, M, x)_j = \sum_{v\geq 0}(-1)^v \dim_k \mathrm{Ext}^v_\Lambda(M, S_j)x^{\delta(v)},$$

where $\delta(v)$ is as in Definition 2.1.

In particular, Λ is a d-Koszul algebra if and only if

$$P(\Lambda, x)_{ij} = \sum_{v\geq 0}(-1)^v \dim_k \mathrm{Ext}^v_\Lambda(S_i, S_j)x^{\delta(v)}.$$

The proof is similar to the one for Koszulity in [BGS, Theorem 2.11.1], see Theorem 1.4.2 in [Y], or [Kr, Theorem 3.2].

We remark that one can also define certain Grothendieck ring valued Hilbert series for Koszul algebras, which is used by Hai and Lorenz [HL] to prove the MacMahon Master Theorem. The idea is as follows.

For simplicity, we assume at this moment that Λ is a connected Koszul algebra. Let H be a bialgebra over k, then one has the tensor product of two H-modules via comultiplication, and hence the Grothendieck group of $\mathrm{mod}(H)$, denoted by $\mathcal{K}$, is a ring with multiplication given by the tensor product. Assume that Λ_i and K_i^i are H-modules for each $i \geq 0$, and all the differential maps $\Lambda \otimes K_i^i \to \Lambda \otimes K_{i-1}^{i-1}$ are H-morphisms, where K^i is as in Section 3. Define the **$\mathcal{K}$-valued Hilbert series** of Λ by $H(\Lambda, \mathcal{K}, x) = \sum_{i \geq 0} [\Lambda_i] x^i \in \mathcal{K}[[x]]$, and the **$\mathcal{K}$-valued Poincaré series** by $P(\Lambda, \mathcal{K}, x) = \sum_{i \geq 0} [K_i^i] x^i$, where $[\Lambda_i]$ and $[K_i^i]$ denote the corresponding elements in $\mathcal{K}$. Then we also have $H(\Lambda, \mathcal{K}, x) P(\Lambda, \mathcal{K}, -x) = 1$. Taking $H = k$ we obtain the Hilbert and the Poincaré series for Koszul algebras.

This kind of Hilbert series works for general homogeneous algebras under some mild conditions. In fact, Etingof and Pak [EP] applied this idea to d-Koszul algebras to obtain certain generalized MacMahon Master Theorem (see also [KP]). More recently, Hai, Kriegk and Lorenz [HKL] applied this kind of Hilbert series to a N-homogeneous superalgebra as well.

5. Dual algebras and Ext-algebras

Let $\Lambda = T_{\Lambda_0}(\Lambda_1)/I$ be a d-homogeneous algebra and $R = I \cap \Lambda_1^{\otimes d}$. For $M, N \in \mathrm{mod}(\Lambda_0^e)$, the dual space $M^* = \mathrm{Hom}_k(M, k)$ is naturally a Λ_0-Λ_0-bimodule; and $(M \otimes N)^*$ is naturally identified with $N^* \otimes M^*$ via $(g \otimes f)(m \otimes n) = f(m)g(n)$ for $f \in M^*, g \in N^*, m \in M$ and $n \in N$. Thus $(\Lambda_1^{\otimes n})^*$ is identified with $(\Lambda_1^*)^{\otimes n}$, $\forall\ n \geq 0$. Set $R^{\perp} = \{f \in (\Lambda_1^*)^{\otimes d} \mid f(r) = 0, \forall\ r \in R\}$. **The dual algebra** of Λ, denoted by $\Lambda^!$, is by definition $T_{\Lambda_0}(\Lambda_1^*)/\langle R^{\perp} \rangle$. Then $(R^{\perp})^{\perp} = R$ if we identify $((\Lambda_1^{\otimes n})^*)^*$ with $\Lambda_1^{\otimes n}$ via the evaluation map, and $(\Lambda^!)^! \cong \Lambda$.

Let $d \geq 2$ be an integer. Put $\mathcal{D} = d\mathbb{N} \cup (d\mathbb{N} + 1)$. By definition $\Lambda_{\mathcal{D}}$ is a positively graded algebra with $(\Lambda_{\mathcal{D}})_i = \Lambda_{\delta(i)}$ for all $i \geq 0$, where $\delta(i)$ is as in Definition 2.1. Note that if $d = 2$ then $\Lambda_{\mathcal{D}} = \Lambda$. We may define a modified Poincaré series $H(\Lambda_{\mathcal{D}}, -1, x)$ of Λ by setting

$$H(\Lambda_{\mathcal{D}}, -1, x)_{ij} = \sum_{v \geq 0} (-1)^v \dim_k(e_i \Lambda_v e_j) x^{\delta(v)}.$$

Note that for a Koszul algebra Λ, $\mathrm{E}(\Lambda)$ is again Koszul and $\mathrm{E}(\Lambda) \cong (\Lambda^!)^{op}$; and hence $\mathrm{E}(\mathrm{E}(\Lambda)) \cong \Lambda$. See [BGS], Theorems 2.10.1 and 2.10.2. This is what Koszul duality means in the level of algebraic structures. For higher Koszulity the situation seems to be not yet well understood, and is worthwhile to be studied (one may look at [Ber1] and [MM] for some related information). However, we have the following characterization of d-Koszulity, see [GMMZ, Section 9], or [BM, Proposition 3.1].

Theorem 5.1. *Keep the above notation. Then a d-homogeneous algebra Λ is d-Koszul if and only if $\mathrm{E}(\Lambda) \cong (\Lambda^!_{\mathcal{D}})^{\mathrm{op}}$ as graded algebras, where $\Lambda^!_{\mathcal{D}}$ is the positively graded algebra with grading $(\Lambda^!_{\mathcal{D}})_i = \Lambda^!_{\delta(i)}$ for $i \geq 0$.*

Together with Theorem 4.3, we have the following consequence.

Corollary 5.2. *Let Λ be a d-Koszul algebra. Then $H(\Lambda, x)H(\Lambda^!_{\mathcal{D}}, -1, x) = I_n$.*

This is a generalization of the quadratic case in [BGS, Lemma 2.11.1]. The converse statement of the corollary is not true, even in the quadratic case. Counterexamples can be found in [Pos2] and [Pion].

Another important consequence is that the Ext-algebra of a d-Koszul algebra is finitely generated. A classical result says that a quadratic algebra is Koszul if and only if its Ext-algebra is generated in degrees 0 and 1, see for example [Lö]. We have the following

Theorem 5.3. ([GMMZ, Theorem 4.1]) *Let Λ be a d-homogeneous algebra. Then Λ is d-Koszul if and only if $\mathrm{E}(\Lambda)$ is generated in degrees 0, 1 and 2.*

Remark 5.4. *It is known that $\mathrm{E}(\Lambda)$ carries a natural A_∞ structure, from which we recover the algebra itself. We refer to* [K] *and* [LPWZ] *for the details. Keller showed that a quadratic algebra Λ is Koszul if and only if the higher (≥ 3) multiplications on $\mathrm{E}(\Lambda)$ are trivial.*

He and Lu ([HLu]) *introduced the so-called $(2, d)$-algebras, and used it to characterize the A_∞-structure on the Ext-algebra $\mathrm{E}(\Lambda)$ of a d-Koszul algebra.*

6. Generalized d-Koszul modules

An interesting topic is to study possible Koszul strucutres arising from a d-Koszul algebra. Let Λ be a d-Koszul algebra. Regrading the Ext-algebra $\mathrm{E}(\Lambda)$, we get a new positively graded algebra $\widehat{\mathrm{E}}(\Lambda)$ with $\widehat{\mathrm{E}}(\Lambda)_0 = \mathrm{E}(\Lambda)_0$, and $\widehat{\mathrm{E}}(\Lambda)_i = \mathrm{E}(\Lambda)_{2i-1} \oplus \mathrm{E}(\Lambda)_{2i}$ for $i \geq 1$. Similarly, for a d-Koszul module M, we define $\widehat{\mathscr{E}}(M)$ by $\widehat{\mathscr{E}}(M)_0 = \mathscr{E}(M)_0$ and $\widehat{\mathscr{E}}(M)_i = \mathscr{E}(M)_{2i-1} \oplus \mathscr{E}(M)_{2i}$ for $i \geq 1$.

Theorem 6.1. ([GMMZ]) *Let Λ be a d-Koszul algebra and M a d-Koszul module. Then*

(*i*) $\mathrm{E}^{\mathrm{ev}}(\Lambda)$ *is a Koszul algebra; and* $\mathscr{E}^{\mathrm{ev}}(M)$ *is a Koszul* $\mathrm{E}^{\mathrm{ev}}(\Lambda)$*-module;*
(*ii*) $\widehat{\mathrm{E}}(\Lambda)$ *is a Koszul algebra; and* $\widehat{\mathscr{E}}(M)$ *is a Koszul* $\widehat{\mathrm{E}}(\Lambda)$*-module.*

An open problem is raised in [GMMZ]: is $\mathscr{E}^{\mathrm{odd}}(M)$ a Koszul $\mathrm{E}^{\mathrm{ev}}(\Lambda)$-module? Marcos and Martínez-Villa [MM] showed that the answer is yes if $\Lambda^!$ is also d-Koszul. However, $\Lambda^!$ is not d-Koszul in general (see [MM], Example 2). Recently, an affirmative answer to this problem is given by Bian, Ye and Zhang, by introducing the so-called generalized d-Koszul modules.

Definition 6.2. ([BYZ]) *Let Λ be a standardly graded algebra, and $d \geq 2$ an integer. A finitely generated graded Λ-module M is a* ***generalized d-Koszul module*** *if M admits a graded projective resolution* (1) *such that each Q^i is generated in degrees in $\Delta(i)$, where*

$$\Delta(i) := \begin{cases} nd, & \text{if } i = 2n, \\ \{nd+1, \cdots, nd+d-1\}, & \text{if } i = 2n+1. \end{cases}$$

If the trivial Λ-module Λ_0 is a generalized d-Koszul module, then we call Λ a ***generalized d-Koszul algebra****.*

As in d-Koszul case we have

Remark 6.3. (*i*) *A graded projective resolution* $\mathbf{Q}^\bullet$ *of a generalized d-Koszul module is unique up to isomorphism; and each Q^i in* (1) *is finitely generated.*

(*ii*) *If M is a generalized d-Koszul module, then a graded projective resolution* $\mathbf{Q}^\bullet$ *of M is minimal, and each syzygy $\Omega^i(M)$ is a graded Λ-module finitely generated in degrees in $\Delta(i)$. In particular, M is finitely generated in degree* 0*.*

(*iii*) *A finitely generated graded Λ-module M is generalized d-Koszul if and only if $\mathscr{E}^{2n}(M)$ is concentrated in degree nd, and $\mathscr{E}^{2n+1}(M)$ is concentrated in degrees in $\{nd+1, \cdots, nd+d-1\}$ for each $n \geq 0$, both with the shift grading.*

It is clear that d-Koszul modules M and the shifts of their syzygies $(\Omega^i M)(-\delta(i))$ are generalized d-Koszul, where $\delta(i)$ is as in Definition 2.1. If M is a generalized d-Koszul Λ-module, then so is the shift $(J^i(M))(-i)$ for each $i \geq 1$, where J is the graded Jacobson radical. Note that a generalized d-Koszul module is not necessarily a d-Koszul module (see Example 2.3 in [BYZ]). As in the d-Koszul case, we have

Theorem 6.4. ([BYZ]) *Let Λ be a d-Koszul algebra, and M a generalized d-Koszul module. Then $\mathscr{E}^{\mathrm{ev}}(M)$ is a Koszul $\mathrm{E}^{\mathrm{ev}}(\Lambda)$-module.*

Since $(\Omega M)(-1)$ is a generalized d-Koszul module, it follows from the above theorem that $\mathrm{E}^{\mathrm{ev}}((\Omega M)(-1))$ is a Koszul $\mathrm{E}^{\mathrm{ev}}(\Lambda)$-module. Therefore by the isomorphism $\mathrm{E}^{\mathrm{odd}}(M) \cong \mathrm{E}^{\mathrm{ev}}(\Omega M) = \mathrm{E}^{\mathrm{ev}}((\Omega M)(-1))$ we have the following consequence, which affirmatively answers the open problem mentioned.

Corollary 6.5. ([BYZ]) *Let Λ be a d-Koszul algebra, and M a d-Koszul Λ-module. Then $\mathrm{E}^{\mathrm{odd}}(M)$ is a Koszul module over the Koszul algebra $\mathrm{E}^{\mathrm{ev}}(\Lambda)$.*

We remark that properties of generalized d-Koszul modules are worthwhile to be studied from various viewpoints, say from the categorical point of view.

7. Lattice Distributivity and Kozulity

Backelin and Fröberg [BF] observe that Koszulity closely links to the lattice distributivity of subspaces. Let V be a k-vector space and $\mathcal{L}(V)$ the set of all subspaces of V. Ordered by inclusion, $\mathcal{L}(V)$ is a lattice with the usual intersection $\cap$ and sum $+$. The lattice $\mathcal{L}(V)$ is **modular** in the sense that $A + (B \cap C) = B \cap (A + C)$ for $A, B, C \in \mathcal{L}(V)$ and $A \subseteq B$. A sublattice of $\mathcal{L}(V)$ is a subset of $\mathcal{L}(V)$ which is closed under taking intersections and sums. The sublattice $\mathcal{L}_S$ generated by a set $S = \{A_1, \cdots, A_n\}$ of subspaces is the smallest sublattice containing $A_1, \cdots, A_n$. A lattice $\mathcal{L}$ is **distributive** if $A \cap (B + C) = (A \cap B) + (A \cap C)$ for $A, B, C \in \mathcal{L}$. We include the following useful criterion.

Lemma 7.1. ([BF]) *Let $S = \{A_1, \cdots, A_n\} \subseteq \mathcal{L}(V)$. Then $\mathcal{L}_S$ is distributive if and only if there is a basis X of V, such that $A_i \cap X$ is a basis of A_i for each i.*

Let $\Lambda = T_{\Lambda_0}(\Lambda_1)/I$ be d-homogeneous and $R = I \cap \Lambda_1^{\otimes d}$. In the following we will omit the symbol "$\otimes$", and write $\Lambda_1^{\otimes n}$ as Λ_1^n, and $\Lambda_1 \otimes R$ as $\Lambda_1 R$. Put $\mathcal{I}_n = \{\Lambda_1^i R \Lambda_1^{n-d-i}\}_{i=0}^{n-d}$ for each $n \geq d$, and $I_n = I \cap \Lambda_1^n$. Clearly $I_0 = I_1 = \cdots = I_{d-1} = 0$, and

$$I_n = \sum_{W \in \mathcal{I}_n} W = R\Lambda_1^{n-d} + \Lambda_1 R \Lambda_1^{n-d-1} + \cdots + \Lambda_1^{n-d} R$$

for $n \geq d$. Then one has

Theorem 7.2. ([Bac], [BGS]) *Keep above notations. Let Λ be a quadratic algebra. Then Λ is Koszul if and only if for each $n \geq 3$, the sublattice $\mathcal{L}_n$ of $\mathcal{L}(\Lambda_1^n)$ generated by $\mathcal{I}_n$ is distributive.*

The case $d \geq 3$ is a little complicated. By Lemma 3.3, if a d-homogeneous algebra Λ is d-Koszul, then

$$\Lambda_1^n R \cap R\Lambda_1^n = \bigcap_{0 \leq i \leq n} \Lambda_1^i R \Lambda_1^{n-i}, \ \forall\ 1 \leq n \leq d-1, \qquad (ol)$$

and we will call this the **overlap condition**. In [Ber1] the following **extra condition** for a d-homogeneous algebra is introduced:

$$\Lambda_1^{d-1} R \cap (\Lambda_1^{d-2} R \Lambda_1 + \Lambda_1^{d-3} R \Lambda_1^2 + \cdots + R\Lambda_1^{d-1}) \subseteq \Lambda_1^{d-2}(R\Lambda_1 \cap \Lambda_1 R). \qquad (ec)$$

The both conditions (ol) and (ec) hold automatically if $d = 2$. It is well-known that the condition (ec) implies the condition (ol), and the d-Koszulity implies (ec) and (ol). A d-homogeneous algebra Λ is said to be **distributive** if the sublattices $\mathcal{L}_n$ are distributive for all $n \geq d+2$. In case Λ is distributive, the condition (ec) is equivalent to the condition (ol). The following result generalizes Backelin's result.

Theorem 7.3. ([Ber3, Proposition 2.3], [Y, Theorem 2.5.1]) *Let Λ be d-homogeneous and distributive. Then Λ is d-Koszul if and only if the condition (ec) is satisfied, if and only if the condition (ol) is satisfied.*

Remark 7.4. *Unlike the quadratic case, it is not known yet whether a d-Koszul algebra is always a distributive d-Koszul for $d \geq 3$, although there is some evidence that this is the case. Fortunately, most d-Koszul algebras of interest are distributive.*

A much more practical condition is Bergman's X-confluence ([Berg]) with respect to a basis X of a vector space V. This confluence condition is stronger than distributivity condition, see for example [Ber1, Theorem 3.11]. An easy example of X-confluence is the case that all relations are monomial, and hence monomial d-homogeneous algebras are always distributive, this fact can also be shown by applying Lemma 7.1 directly. We thus have the equivalence between d-Koszulity and the condition (ol) in this case, see [Ber1, Proposition 3.8], and [YZ2] for the nonconnected case. In fact, we have the following more general result.

Theorem 7.5. ([YZ2, Theorem 2.1]) *Let $d \geq 2$ and Λ a monomial d-homogeneous algebra. Then the following are equivalent:*

- (i) Λ *is d-Koszul;*
- (ii) Λ *satisfies the condition (ol);*
- (iii) $\mathrm{GExt}^3_\Lambda(\Lambda_0, \Lambda_0(i)) = 0$ *unless $i = d + 1$;*
- (iv) *Any linearly presented Λ-module is a d-Koszul module.*

In the quadratic case, the condition (ol) satisfies automatically and hence all monomial quadratic algebras are Koszul, which is a classical result on Koszul algebras.

Another example by using X-influence is as follows. Let V be a finite-dimensional k-space. Let $d \geq 3$ and denote by $\mathfrak{S}_d$ the symmetric group of $\{1, 2, \cdots, d\}$. Set $\bigwedge^d V$ to be the k-span of

$$\left\{ \sum_{\sigma \in \mathfrak{S}_d} sgn(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(d)} \mid a_1 \otimes \cdots \otimes a_d \in V^{\otimes d} \right\}.$$

If the characteristic of k is 0, then $TV/\langle \bigwedge^d V \rangle$ is d-Koszul ([Ber1, Theorem 3.13]).

8. More related topics

We survey more related topics with d-Koszul algebras.

8.1. **AS-Gorenstein property.** An algebra is **left Gorenstein** if ${}_AA$ is of finite injective dimension. A connected standardly graded k-algebra Λ is **left AS-Gorenstein** if the injective dimension r of ${}_\Lambda\Lambda$ is finite, and $\dim_k \mathrm{Ext}^r_\Lambda(k, \Lambda) = 1$ and $\dim_k \mathrm{Ext}^i_\Lambda(k, \Lambda) = 0$ for $i \neq r$. If in addition, Λ is of finite global dimension D, then Λ is said to be **left AS-regular**; and in this case one has $D = r$. Notice that left AS-regularity implies right AS-regularity ([SZ], Proposition 3.1). In lower dimensional case, the AS-Gorenstein property links to higher Koszulity closely.

In fact, let Λ be a AS-regular algebra with global dimension $D = 2$ or 3. Then Λ is d-Koszul with $d = 2$ if $D = 2$, and $d \geq 2$ if $D = 3$. See [BM, Proposition 5.2]. In the same work, Berger and Marconnet prove that a Koszul algebra Λ of finite global dimension is AS-Gorenstein if and only if $\mathrm{E}(\Lambda)$ is Frobenius (see Theorem 5.4 and Corollary 5.12 in [BM]). Moreover, [BM, Theorem 6.3] claims that for an AS-Gorenstein d-Koszul algebra of finite global dimension, then there is a Poincaré duality between its Hochschild homology and cohomology, which relates the Van den Bergh duality [V1, Proposition 2].

Mao and Wu studied the AS-regular property of higher Koszul algebras. Let Λ be a left Gorenstein d-Koszul algebra which is left Noetherian. Then Λ is AS-regular if and only if it is d-standard (see [MW, Theorem B] for details).

Dubois-Violette ([D1], [D2]) considered homogeneous algebras defined by multilinear forms. It was shown that a Gorenstein d-Koszul algebra of finite global dimension is defined by some multilinear form ([D2], Theorem 5 and Theorem 11). An important example of such algebras are Yang-Mills algebras, which are 3-Koszul algebra of global dimension 3 and are Gorenstein, see [CD].

8.2. **PBW deformations.** PBW deformations for quadratic algebras are studied by Braverman and Gaitsgory in [BrG]. Fløystad and Vatne considered general homogeneous case, and gave a criteria (the Jacobi condition) for PBW-deformations ([FV], Theorem 1.1). This is also obtained by Berger and Ginzburg [BG] independently.

In [Pos1], Positselski showed that the PBW deformations of a quadratic Koszul algebra correspond to the differential graded structures on its Ext-algebra. Inspired by this correspondence, Fløystad and Vatne ([FV, Theorem 2.1]) showed that for a d-Koszul algebra, the PBW deformations correspond to certain class of A_∞ structures on its Ext-algebra.

Cassidy and Shelton ([CS1]) characterized PBW deformations for an graded algebra Λ by using the notion of central extensions and certain homological constant attached to Λ.

An interesting application is to the study of PBW deformations of certain Calabi-Yau algebras. It is conjectured by Van den Bergh [V2] that any graded quiver algebra which is Calabi-Yau of dimension 3 is defined from a super potential. This is proved in the graded situation by Bocklandt [Boc] (see also [BSW]). A key observation by Berger and Taillefer is that such algebras are d-Koszul; and they obtain a PBW-deformation Λ' of a graded quiver algebra which is Calabi-Yau of dimension 3, and prove that Λ' is again Calabi-Yau. See Theorem 1.1 in [BT].

8.3. **Generalizations of Koszulity.** There are quite a lot of generalizations of Koszulity from different perspective.

One aspect is to consider graded algebra Λ such that $\mathrm{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$ is concentrated in certain degrees for each $i \geq 0$. In this sense, one has the notion of δ-Koszul algebras and 2-d-determined algebras introduced by Green and Marcos [GM1, GM2], piecewise Koszul algebras by Lü, He and Lu [LHL], and bi-Koszul algebra by Lu and Si [LS1, LS2, S].

Another one is to consider the Ext-algebra. Recall that a quadratic algebra is Koszul if and only if its Ext-algebra is generated in degrees 0 and 1; and that a d-homogeneous algebra ($d \geq 2$) is d-Koszul if and only if its Ext-algebra is generated in degrees 0, 1 and 2. A graded algebra such that the Ext-algebra is generated in degrees $0, 1$ and 2 is called a $\mathcal{K}_2$-algebra in [CS2]; and the Ext-algebra of a $\mathcal{K}_2$-algebra may not be $\mathcal{K}_2$ ([CPS]).

The Ext-algebra of a δ-Koszul algebra in the sense of [GM1] has also been studied by various authors. It was asked in [GM1] whether there exists a common bound N, such that for any δ-Koszul algebra, the Ext-algebra is generated in degrees $\{0, 1, \cdots, N\}$. Recently Lü [Lü] gave an negative answer to this question. An example of quadratic δ-Koszul algebra whose Ext-algebra is exactly generated in two different degrees was given in [C].

8.4. **Generalized Koszul Dualality.** It is well-known that there exists a Koszul duality between a Koszul algebra Λ and its Ext-algebra $\mathrm{E}(\Lambda)$ via the quadratic dual, as well as the equivalence of bounded derived categories ([BGS]). While for d-Koszul algebras, very little is known about this, which seems to be of importance and interest. We mention that in [LPWZ] some results in this direction for non-Koszul case, including d-Koszul algebras, are discussed by using A_∞ algebras.

We have seen in Section 5 that for a d-Koszul algebra Λ, $\mathrm{E}(\Lambda)$ can be obtained from its dual algebra. In a series work [MS1, MS2, MS3], Martínez-Villa and Saorín made an attempt to connect the module category of a d-Koszul algebra Λ and the one of its Ext-algebra $\mathrm{E}(\Lambda)$. They showed that there is a subcategory of $\mathrm{Gr}(\mathrm{E}(\Lambda))$ that embeds fully faithfully in the category of cochain complexes of graded Λ-modules.

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Yu Ye: Department of Mathematics, University of Science and Technology of China, Hefei 230026, P. R. China

Pu Zhang: Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, P. R. China